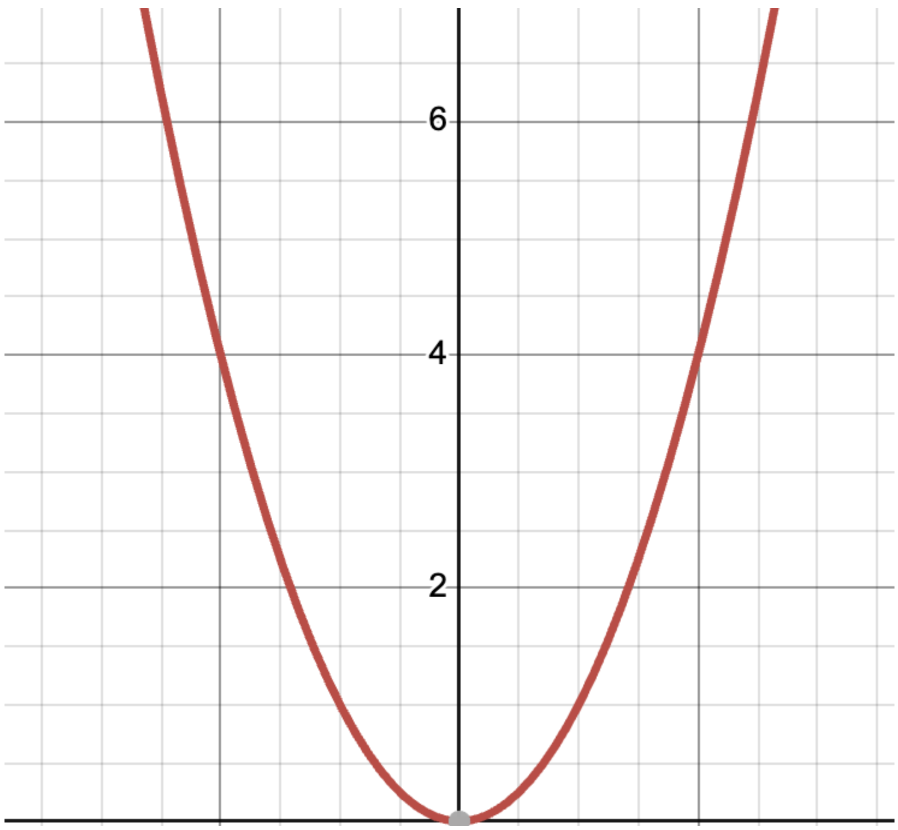
In this blog post, I will share my thoughts around the key ideas of Calculus.

Average Rate of Change

Assume that you have a function $y = x^2$ as shown below.



What is the rate of change between 2 points (x, x + h). Well, to compute rate of change (or) slope between 2 points, we know that we need to compute rise over run (change in y over change in x):

Average Rate of Change = change in y / change in x = f(x+h) - f(x) / h

For the points, between say x = 2 and x = 4, slope or rate of change = 12/2 = 6. Slope for any 2 points in this function x^2 keeps changing and it is not constant. This gives the **average** rate of change between 2 points.

Instantaneous Rate of Change

From the above, we know how to find the average rate of change between 2 points which can be described as: f(x+h) - f(x) / h. What if we need to find the instantaneous rate of change at a given point? The challenge to this problem is, since we have a single point, the denominator h will become zero and the overall slope function will be undefined. So, how do we find the instantaneous rate of change at a given point, as the slope function turns undefined? Well, let us explore the idea of limits.

Limits

The idea of limits is very simple but yet very profound. For the same function $y = x^2$, and at x = 3, we know that the function value is y = 9. Now, instead of figuring out the function value at x = 3, what is the value of the function **when x approaches 3**. This is the core idea of limits. Say for eg, we start at a point, say x = 2.5, and see what happens to the function $y = x^2$, as we are approaching from x = 2.5 to x = 3.

x	**$y = x^2$**
2.5	6.25
2.75	7.56
2.95	8.7
2.99	9.84
2.9999	8.9994
2.99999999	**8.99999994**

We can do a similar thing of approaching 3 from the other direction, say from 3.5 => 3.25 => 3.05 => 3.001 => 3.00001 => 3.000000001. In both cases, we see that the limit at $x \to 3$ for function x^2 is 9. We write this as:

$$\lim_{x \to 3} x^2 = 9$$

Limits is one of the key ideas of Calculus. It lets us examine what happens to a function when we approach a certain value.

Figuring out Instantaneous Rate of Change

Now with the idea of limits, we will try to find the instantaneous rate of change. Key intuition is, we start with 2 points as we did to find average rate of change, x and x + h and we start sliding x + h towards x by bringing h closer to 0. h cannot be equal to zero as it will make the slope function undefined, but it is so small that there is not much difference between the value of x and x+h, which will help us find the slope of a curve at a point. We know that the average rate of change is f(x+h) - f(x) / h, now we will try to find the slope by applying the lim of $h \to 0$, which will equate to :

Instantaneous rate of change for y = f(x) $= \lim_{h\to 0} f(x + h) - f(x) / h$

If we can solve for the above function and eliminate h from the denominator we will be able to find the instantaneous rate of change at a given point.

For $y = x^2$, this will translate to,

Instantaneous rate of change for $y = x^2 = \lim_{h\to 0} ((x + h)^2 - x^2)/ h$

Instantaneous rate of change for $y = x^2 = \lim_{h\to 0} ((x^2 + h^2 + 2xh - x^2)/ h$

Instantaneous rate of change for $y = x^2 = \lim_{h\to 0} ((h^2 + 2xh)/ h$

Instantaneous rate of change for $y = x^2 = \lim_{h\to 0} (h(h + 2x))/ h$

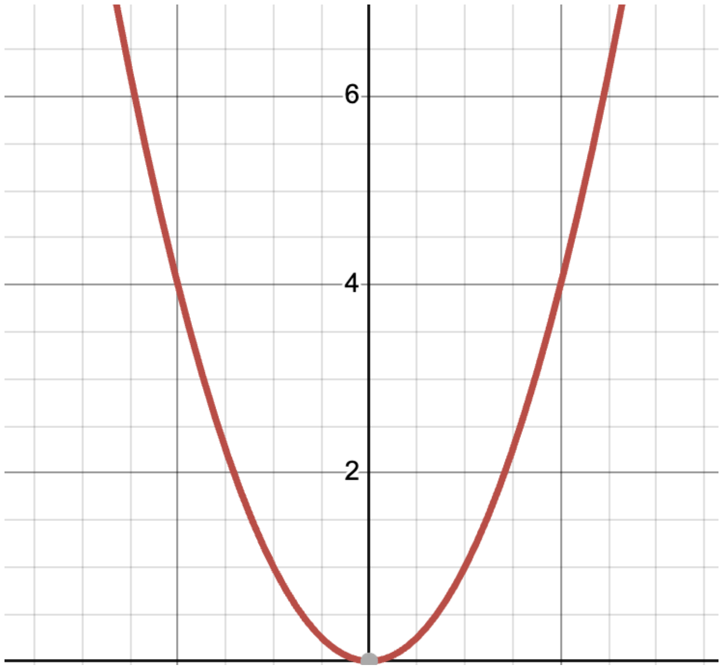
Instantaneous rate of change for $y = x^2 = \lim_{h\to 0} (h + 2x)$

If we apply limit $h\to 0$, we have
Instantaneous rate of change for $y = x^2 = 2x$

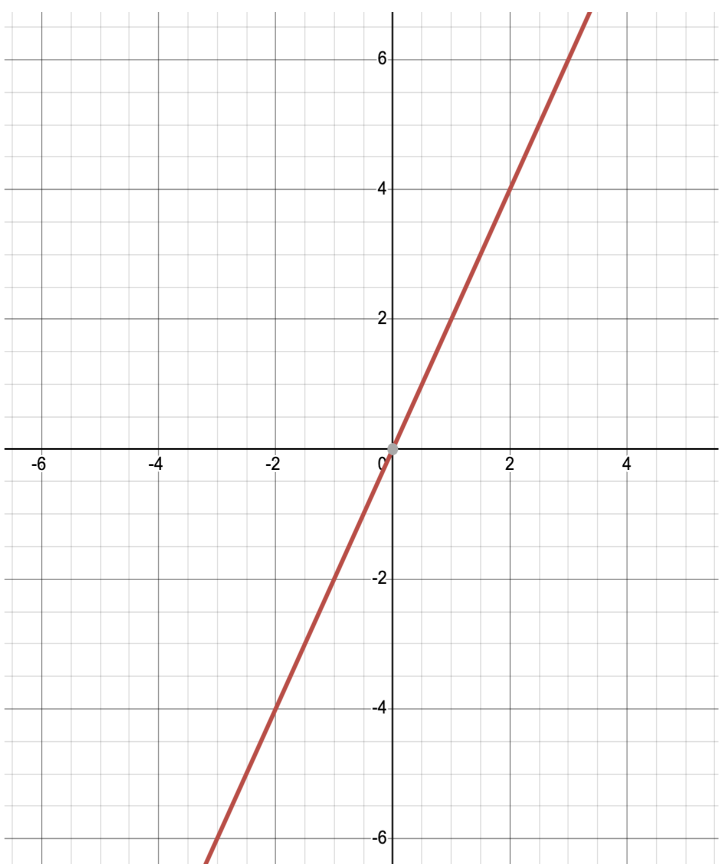
The instantaneous rate of change of a function is called **"derivative"**. Derivatives (or) instantaneous rate of change or slope of a curve at any given point of a function is the second key idea of Calculus.

To sum things up, we have 2 functions now, the original function which we started with which is $y = x^2$ and the derivative of function $y = x^2$ which is equal to 2x

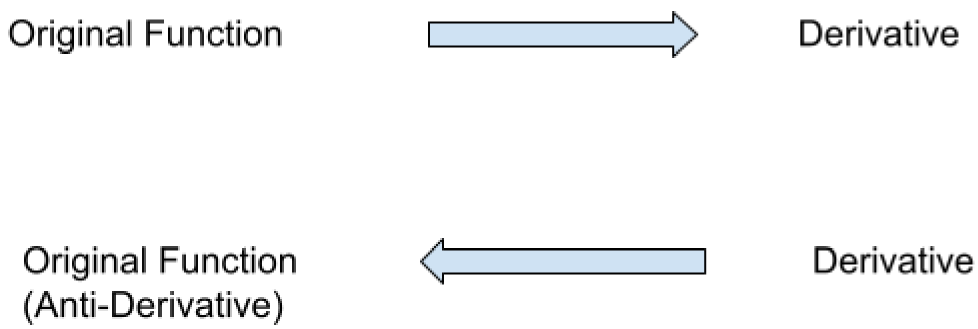
Original function $y = x^2$



Derivative of function $y = x^2$ is equal to $2x$



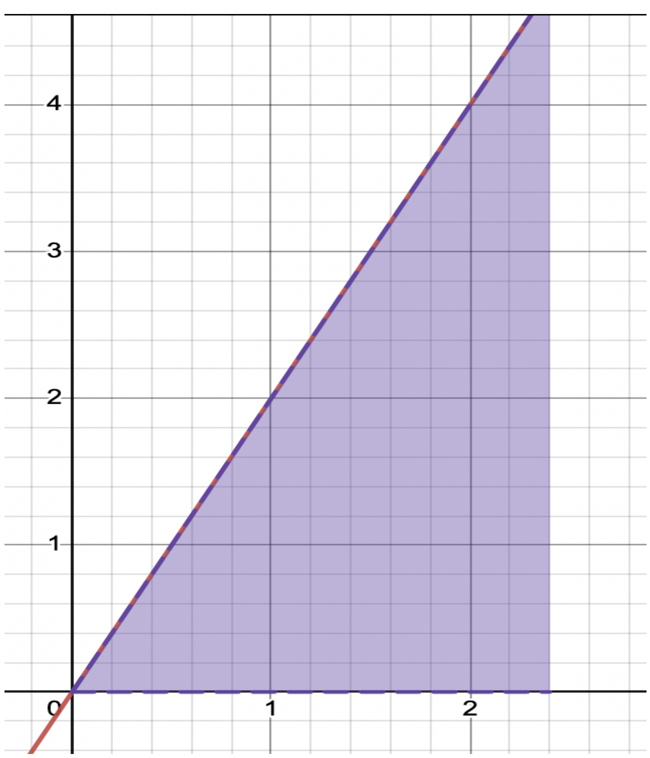
In this example, we started off with a function y = x^2 and we got a brand new function 2x which was its derivative. Now, if we had started off with a function y = 2x, and if we had assumed that this is the instantaneous rate of change for another function, what should that original function be? This is called finding anti-derivative or figuring out the original function whose derivative is the provided function.



Ok, we assume that the given function is a derivative of some other function, and we find the anti-derivative to figure out the original function, so what is the point in doing it? Well, to understand its implications, let's explore how we can identify the second interesting problem using Calculus: Finding area under the curve.

Area under the curve

Suppose, we start with the function y = 2x and we are trying to find the area under the curve for this function as shown below:



How do we go about solving the problem? Again, we will use our good old friend, limits. Assume that the area under the curve can be broken down into a series of rectangles. We know that the area of a rectangle can be found using height x width. This should give us a reasonable approximation for finding the area under the curve. To move from an approximate area to an accurate area, what if we create an infinite number of infinitesimally small rectangles and pack them so tightly by applying the limit of width → 0, and then do a summation of all of those rectangles. This summation should give an accurate area under the curve for the given function. With the limit of width → 0, height of that rectangle becomes the value of the function at the starting point. This idea can be extended not just for function y = 2x, but for any function f(x). To generalize this idea, area under the curve for a function f(x), can be defined as,

One infinitesimally small rectangle area, $dF = \lim_{dx \to 0} f(x) \cdot dx$,

where dF = infinitesimally small area and dx = infinitesimally small width.

Summation of all infinitesimally small rectangle $= \int f(x) \cdot dx = F(x)$

gives area under the curve. The symbol ∫ is called **integration**, or finding area under the curve for a function f(x). Integration (or) finding the area under the curve is the third key idea of Calculus. If we stop to think about it, for the function y = 2x, we could have computed the area easily as the area under the graph is bound by the shape of the triangle, whose area can be computed as ½ * base * height. But for most of the functions, area under the graph do not conform to known geometric shapes.

In the above integral, we introduced a new function F(x). We know it represents the area under the graph for our function f(x), but we don't know what it truly represents. Is there any connection between the function f(x) and F(x)? The fundamental theorem of Calculus - I answer this question beautifully. Before we look into it, one intuition which we can see in the above examples of function $f(x) = x^2$ and f(x) = 2x, while covering derivatives and integrals are:

Under derivatives:
Instantaneous rate of change, which was 2x
= infinitesimally small change for **function x^2** / infinitesimally small change of x, by applying limit h → 0

Under integration:
Area of an infinitesimally small rectangle under the curve 2x
= Value of function 2x at the starting point * infinitesimal small width, by applying limit width → 0

Can we see if there is a connection between **function x^2** and **Area under the curve for function 2x**? The connection is provided by the fundamental theorem of calculus I.

Fundamental theorem of Calculus - I

If f(x) is continuous over an interval [a, b], and the function F(x) is defined by, F(x) = $\int_{a}^{x}$ f(t) . dt,

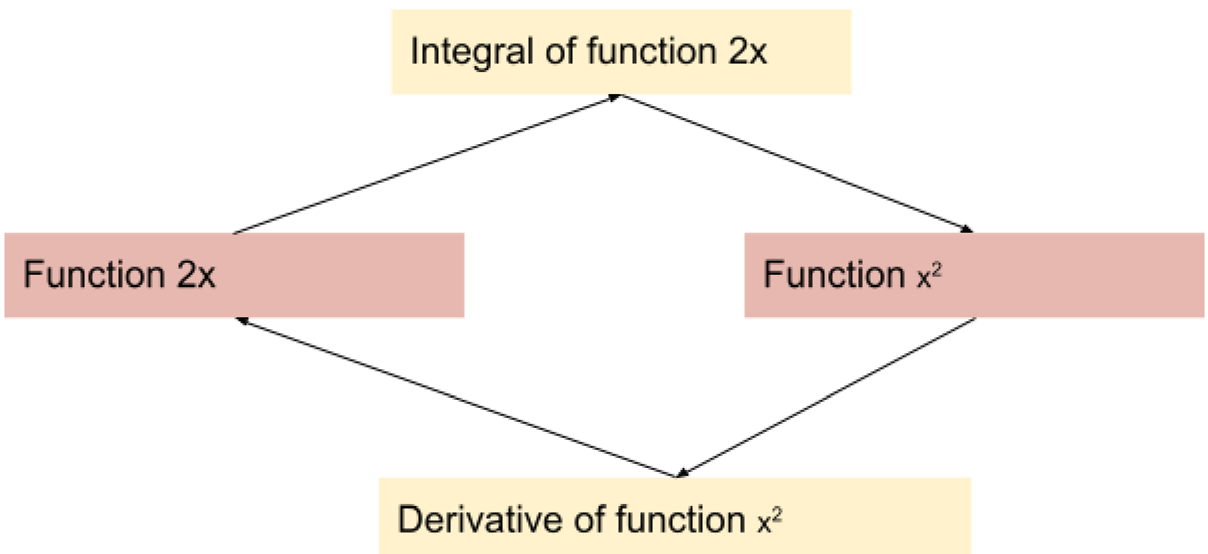
then F'(x) = f(x) over [a,b]

Another way of stating the above theorem is,

$$\frac{d}{dx}\int_{a}^{x} f(t)\,.\,dt = f(x)$$

which states that the derivative of an integral of a function gives us back the original function.

In the above example, if we take the integral of function 2x, we will get the function x^2 which, if we differentiate, gives back the original function 2x. This can be shown in the below diagram:



Fundamental theorem of Calculus - II

If f is continuous over the interval [a, b], and F(x) is any antiderivative of f(x), then

$$\int_{a}^{b} f(x)\,.\,dx = F(b) - F(a)$$

Another way of stating the above theorem is,

$\int_{a}^{b}$ f '(x) . dx = f(b) - f(a), where f '(x) is the derivative of f(x)

which states that, ***when you integrate a derivative, you get the total change in quantity.***

The beauty of this theorem is the simplicity by which we can compute the area under a curve for a given function. To find the area under a curve between an upper and lower bound for a given function, all we have to do is to find the corresponding antiderivative for that function and compute the value of the antiderivative function at those bounds and subtract those values. Area under the curve can be viewed as an **accumulation** function.

Conclusion

In Calculus, through the concept of limits, we figured out the instantaneous rate of change at any point in a curve (derivatives) and area under the curve (integration). By doing so, we have understood that there is a deep connection between derivatives and integration, and they both are inverses of each other.